Problem Set I: Preferences, W.A.R.P., consumer choice

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Recap: $\succsim$, $\succ$, $\sim$

Definition

The *strict preference relation* $\succ$ is

$$x \succ y \iff x \succsim y \text{ but not } y \succsim x$$

Definition

The *indifference relation* $\sim$ is

$$x \sim y \iff x \succsim y \text{ and } y \succsim x$$

Recap: $\succsim$ rationality assumptions

$\succsim$ is rational if it is

- Complete: $\forall x, y \in X$, we have $x \succsim y$ or $y \succsim x$ or both;
- Transitive: $\forall x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Recap: $\succsim$ and utility function $u(\cdot)$

Definition

A function $u : X \mapsto \mathbb{R}$ is a utility function representing $\succsim$ if

$$\forall x, y \in X : \quad x \succsim y \iff u(x) \geq u(y)$$

1. MWG, Exercise 1.B.1 + 1.B.2: properties of $\succsim$

Prove that if $\succsim$ is rational (complete and transitive), then

1. $\succ$ is both irreflexive ($x \succ x$ never holds) and transitive (if $x \succ y$ and $y \succ z$, then $x \succ z$);
2. $\sim$ is reflexive ($x \sim x$, $\forall x$), transitive (if $x \sim y$ and $y \sim z$, then $x \sim z$) and symmetric (if $x \sim y$ then $y \sim x$);
3. if $x \succ y \succsim z$ then $x \succ z$.

Solution: property 3 first

Proof.

Property 3: if $x \succ y \succsim z$ then $x \succ z$

1. By definition, $x \succ y$ means that $x \succsim y$ but not $y \succsim x$;
2. then, $x \succ y \succsim z$ means $x \succsim y \succsim z$;
3. for transitivity (assumed), this means that $x \succsim z$.
4. Now, let's suppose that $z \succsim x$. Since $y \succsim z$, by transitivity we'd have $y \succsim x$
5. but this is a *contradiction*, since we had in the beginning that $x \succ y$.
6. So, we have $x \succsim z$ but we cannot have $z \succsim x$: this means that $x \succ z$

□

Solution: property 1

Proof.

Property 1: $\succ$ is irreflexive and transitive

1. **Irreflexivity**. Use completeness: $x \succsim y$, $\forall x, y \in X$:
2. hence, it must hold also for $x \succsim x$, $\forall x \in X$;
3. this means that in no case there can be $x \succ x$.
4. **Transitivity**. Suppose $x \succ y$ and $y \succ z$:
5. this means that at least $x \succ y \succsim z$.
6. But we have proved before that this means $x \succ z$.

□

Solution: property 2

Proof.

Property 2: $\sim$ is reflexive, transitive and symmetric

1. **Reflexivity**. By completeness, $x \succsim x$, $\forall x \in X$:
2. this implies also that $x \sim x$, $\forall x \in X$, by definition of $\sim$.
3. **Transitivity**. Suppose $x \sim y$ and $y \sim z$:
4. by the definition of $\sim$, this means that all of these hold:
5. $x \succsim y, \ y \succsim x, \ y \succsim z, \ z \succsim y$.
6. By transitivity of $\succsim$, this implies both $x \succsim z$ and $z \succsim x$: hence $x \sim z$.
7. **Symmetry**. Suppose $x \sim y$: by definition, then $x \succsim y$ and $y \succsim x$.
8. But the latter is also the definition of $y \sim x$, if you look it the other way around.
9. hence, $x \sim y$ implies $y \sim x$.

□

2. MWG 1.B.3 + 1.B.4.: $\succsim$ and $u(\cdot)$

- Show that if $f : \mathbb{R} \mapsto \mathbb{R}$ is a strictly increasing function and $u : X \mapsto \mathbb{R}$ is a utility function representing the preference relation $\succsim$, then the function $v : X \mapsto \mathbb{R}$ defined by $v(x) = f(u(x))$ is also a utility function representing $\succsim$;
- Consider a preference relation $\succsim$ and a function $u : X \mapsto \mathbb{R}$. Show that if $u(x) = u(y)$ implies $x \sim y$ and if $u(x) > u(y)$ implies $x \succ y$ then $u(\cdot)$ is a utility function representing $\succsim$.

Solution: strictly increasing function, intuition

- We will prove that a utility function associated with $\succsim$ is *ordinal* and not *cardinal* in nature.
- This is important: among other things, it implies that it is impossible to make interpersonal utility comparisons directly.
- Note the definition of strictly increasing function:

Definition

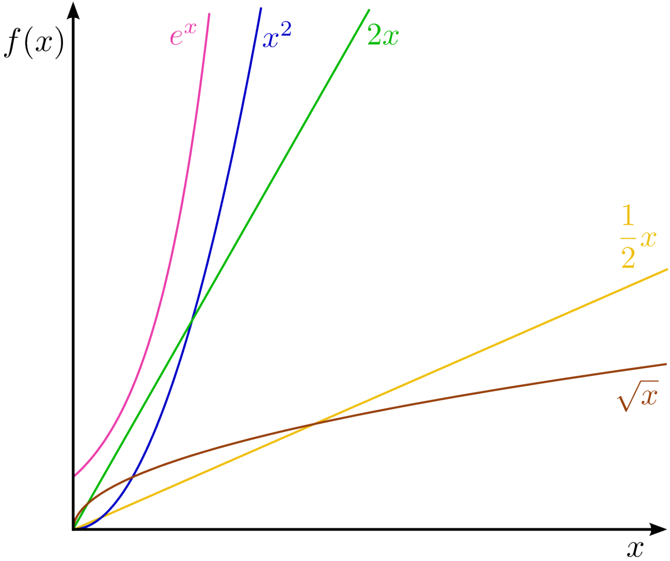
A function $f(x)$ is said to be *strictly increasing* over an interval I if $f(b) > f(a)$ for all $b > a$, when $a, b \in I$.

Example

Functions that are strictly increasing over their whole domain are among others all positive straight lines ($y = ax$, $a > 0$) and positive exponentials ($y = a^x$, $a > 0$); other functions can be increasing over a part of their domain, as parabola ($y = x^2$, for $x > 0$).

- Tip: a strictly increasing function on interval I has its derivative *positive* on I.

Solution: strictly increasing functions, plots



Solution: strictly increasing function, proof

Proof.

A strictly increasing transformation of a utility function is still a utility function

1. Let's take $x, y \in X$. Since $u(\cdot)$ represents $\succsim$, by definition:
2. if $x \succsim y$ then $u(x) \geq u(y)$.
3. since $f(\cdot)$ is strictly increasing, applying $f(\cdot)$ to $u(\cdot)$ does not change order, but only magnitude;
4. hence, $f(u(x)) \geq f(u(y))$, i.e. $v(x) \geq v(y)$ when $x \succsim y$:
5. hence, $v(\cdot)$ is a utility function representing $\succsim$.

□

Solution: $u(\cdot)$, $\succ$ and $\sim$.

Proof.

if $x \succsim y$, then $u(x) \geq u(y)$

1. Suppose $x \succsim y$.
2. if at this we add $y \succsim x$, then $x \sim y$ and $u(x) = u(y)$.
3. if instead we don't have $y \succsim x$, then $x \succ y$ and $u(x) > u(y)$.
4. hence, if $x \succsim y$, then $u(x) \geq u(y)$

□

Proof.

if $u(x) \geq u(y)$, then $x \succsim y$

1. Suppose $u(x) \geq u(y)$.
2. if at this we add $u(x) = u(y)$, then $x \sim y$.
3. if instead we add $u(x) > u(y)$, then $x \succ y$.
4. hence, if $u(x) \geq u(y)$, then $x \succsim y$.

□

3. MWG 1.B.5: $\succsim$ and $u(\cdot)$, II

Show that if X is finite and $\succsim$ is a rational preference relation on X, then there is a utility function $u : X \mapsto \mathbb{R}$ that represents $\succsim$.

Solution: intuition

- Since X is finite, the set of pairwise combination of elements of X is finite too;
- Since $\succsim$ is rational (hence *complete* and *transitive*):
  - it defines a preference over all of the finite set of pairs;
  - it excludes contradictory cycles of preferences.
- Hence, intuitively it is possible to rank all $x, y \in X$ according to $\succsim$;
- it must then be possible to build a utility function with such a complete ranking using $\geq$.

Solution: proving by induction

- A proof by induction is done by showing that something is true for $n = 1$ and then for $n + 1$;
- it then follows that it must be true for all n up to N.
- Proof by induction is used in the set of natural numbers $\mathbb{N}$.

- More formally, for any proposition $P(n)$ about positive integers:
- Prove that $P(1)$ is true (*base case*);
- Prove that for each $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is true (*inductive step*).

Example

Consider a set of domino tiles. If domino tile n falls, tile $n + 1$ will fall. If we prove that tile 1 has fallen, then we can conclude that all tiles will fall.

Solution: proof, $x \nsim y$

Proof.

if X is finite, then there exists a $u(\cdot)$ representing $\succsim$: no indifference

1. Start considering that no two items are indifferent, i.e. $x \nsim y, \forall x, y \in X$;
2. Let's prove by induction that in such a setting there exists a $u(\cdot)$ representing $\succsim$.
3. **Base case:** if $N = 1$ there is nothing to prove.
4. **Inductive step:** Let's suppose the claim is true for $N - 1$, and let's prove it is still ture for N.

1. Let's take $X = \{x_1, x_2, \dots, x_{N-1}, x_N\}$.
2. By hypothesis, there exists a $u(\cdot)$ on $\succsim$ defined up to x_{N-1}.
3. Let's order the x: let's assume $u(x_1) > u(x_2) > \cdots > u(x_{N-1})$.
4. Since we have assumed no indifference, the above ranking means exclusively:

$$\begin{cases} \forall i < N, \; x_N \succ x_i \\ \forall i < N, \; x_i \succ x_N \\ \exists i < N \text{ and } j < N \text{ s.t. } x_i \succ x_N \succ x_j \end{cases}$$

Solution: proof, $x \nsim y$ continued

Proof.

...continued

In all the three cases above we can find a value of $u(\cdot)$ that is consistent:

1. In **Case 1**, we can take $u(N) > u(x_1)$;
2. In **Case 2**, we can take $u(N) < u(x_N - 1)$;
3. in **Case 3**:
   - Define two intervals $I = \{i \in (1 \dots N) : x_i \succ x_N\}$ and $J = \{j \in (1 \dots N) : x_N \succ x_j\}$;
   - I and J are disjoint intervals on $\mathbb{N}$ by our hypotheses;
   - then if $i^* = max\ I$, $i^* + 1 = min\ J$.
   - We can then take $u(x_N)$ to lie in the interval $(u(i^*), u(i^* + 1))$.

Hence, in all of three cases an utility function can be built. □

Recap: W.A.R.P., $\succsim^*$

Definition (WARP)

A choice structure $(\mathcal{B}, C(\cdot))$ satisfies the weak axiom if for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathcal{B}$ with $x, y \in B'$, if we have $y \in C(B')$ we must also have $x \in C(B')$

- Which is indeed a minimal consistency requirement. Note that completeness and transitivity are not required.

Definition (Revealed preference relation $\succsim^*$)

Given a choice structure $(\mathcal{B}, C(\cdot))$, the revealed preference relation $\succsim^*$ is defined as:

$x \succsim^* y \iff$ there is some $B \in \mathcal{B}$ such that $x, y \in B$ and $x \in C(B)$.

- Which is just 'attaching a preference relationship' to choices
- Note again that nor completeness nor transitivity are implied. It is just descriptive.

Recap: Rationalizability

Rational preferences $\succsim$	$\Rightarrow$	W.A.R.P. satisfied	$\checkmark$ always
W.A.R.P. satisfied	$\Rightarrow$	Rational preferences $\succsim$	$\times$ not always

Definition (Rationalizability)

Given a choice structure $(\mathcal{B}, C(\cdot))$, the rational preference relation $\succsim$ rationalizes $C(\cdot)$ relative to $\mathcal{B}$ if $C(B) = C^*(B, \succsim)$ for all $B \in \mathcal{B}$. In other words, $\succsim$ *generates* the choice structure $(\mathcal{B}, C(\cdot))$.

- the W.A.R.P. is a necessary but not sufficient condition for rationalizability.
- if $\mathcal{B}$ includes all subsets of X of up to three elements, then it is also sufficient:
- intuitively, the three-members property implies transitivity...

4. Exercise on W.A.R.P.

Consider a choice problem with choice set $X = \{x, y, z\}$. Consider the following choice structures:

- $(\mathcal{B}', C(\cdot))$, in which $\mathcal{B}' = \{\{x, y\}, \{y, z\}, \{x, z\}, \{x\}, \{y\}, \{z\}\}$ and $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{z\}$, $C(\{x\}) = \{x\}$, $C(\{y\}) = \{y\}$, $C(\{z\}) = \{z\}$.
- $(\mathcal{B}'', C(\cdot))$, in which $\mathcal{B}'' = \{\{x, y, z\}, \{x, y\}, \{y, z\}, \{x, z\}, \{x\}, \{y\}, \{z\}\}$ and $C(\{x, y, z\}) = \{x\}$,
  $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{z\}$, $C(\{x, z\}) = \{z\}$, $C(\{x\}) = \{x\}$, $C(\{y\}) = \{y\}$, $C(\{z\}) = \{z\}$.
- $(\mathcal{B}''', C(\cdot))$, in which $\mathcal{B}''' = \{\{x, y, z\}, \{x, y\}, \{y, z\}, \{x, z\}, \{x\}, \{y\}, \{z\}\}$ and $C(\{x, y, z\}) = \{x\}$,
  $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{x\}$, $C(\{x\}) = \{x\}$, $C(\{y\}) = \{y\}$, $C(\{z\}) = \{z\}$.

For every choice structure say if the WARP is satisfied and if it exists a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to *its* $\mathcal{B}$. If such a rationalization is possible, write it down. Comment on your results.

Solution: $(\mathcal{B}', C(\cdot))$

The choice structure can be summarised in these three relations:

- $C(\{x, y\}) = \{x\}$ reveals $x \succsim^* y$;
- $C(\{y, z\}) = \{y\}$ reveals $y \succsim^* z$
- $C(\{x, z\}) = \{z\}$ reveals $z \succsim^* x$

1. W.A.R.P. is *trivially* satisfied
   - the same couple never appears more than once in different budgets;
   - moreover, $\mathcal{B}'$ does not include all budgets up to three elements.
   - $\succsim^*$ revealed preference relation is not necessarily transitive
2. $\mathcal{B}'$ is NOT rationalizable:
   - $C(\{x, y\}) = \{x\}$ is rationalised by $x \succ y$;
   - $C(\{y, z\}) = \{y\}$ is rationalised by $y \succ z$;
   - $C(\{x, z\}) = \{z\}$ is rationalised by $z \succ x$.
   - It is not transitive, hence $(\mathcal{B}', C(\cdot))$ is not rationalisable.

Solution: $(\mathcal{B}'', C(\cdot))$

The choice structure can be summarised in these relations:

- $C(\{x, y, z\}) = \{x\}$ reveals $x \succsim^* y$ and $x \succsim^* z$
- $C(\{x, y\}) = \{x\}$ reveals $x \succsim^* y$;
- $C(\{y, z\}) = \{y\}$ reveals $y \succsim^* z$;
- $C(\{x, z\}) = \{z\}$ reveals $z \succsim^* x$.

1. W.A.R.P. is NOT satisfied
   - $x \succsim^* z$ and $z \succsim^* x$ hold at the same time;
   - in this case it exists $x, z \in B : C(B) = \{x\}$, but there is also...
   - ...a $x, z \in B' : z \in C(B')$ but not $x \in C(B')$
2. $\mathcal{B}''$ is NOT rationalisable:
   - since in general if $\succsim$ is rational $\Rightarrow \succsim^*$ satisfies W.A.R.P.;
   - then, by using the contrapositive, if $A \Rightarrow B$, it must be true that $\neg B \Rightarrow \neg A$
   - Hence $(\mathcal{B}'', C(\cdot))$ is not rationalisable

Solution: $(\mathcal{B}''', C(\cdot))$

The choice structure can be summarised in these relations:

- $C(\{x, y, z\}) = \{x\}$ reveals $x \succsim^* y$ and $x \succsim^* z$
- $C(\{x, y\}) = \{x\}$ reveals $x \succsim^* y$;
- $C(\{y, z\}) = \{y\}$ reveals $y \succsim^* z$;
- $C(\{x, z\}) = \{x\}$ reveals $x \succsim^* z$.

1. W.A.R.P. is satisfied
   - there are no violations of the type $x \succsim^* y$ and $y \succsim^* x$;
   - moreover, $\mathcal{B}'''$ *includes* all budgets up to three elements.
2. $\mathcal{B}'''$ is rationalizable:
   - $C(\{x, y, z\}) = \{x\}$ reveals $x \succ y$ and $x \succ z$
   - $C(\{x, y\}) = \{x\}$ reveals $x \succ y$;
   - $C(\{y, z\}) = \{y\}$ reveals $y \succ z$;
   - $C(\{x, z\}) = \{x\}$ reveals $x \succ z$.
   - Hence $x \succ y \succ z$ is complete and transitive and rationalises $(\mathcal{B}''', C(\cdot))$

5. MWG 1.D.2: $\succsim$ and W.A.R.P.

Show that if X is finite, then any rational preference relation generates a nonempty choice rule; that is, $C(B) \neq \varnothing$ for any $B \subset X$ with $B \neq \varnothing$.

Solution

Proof.

X finite $\Rightarrow C(B) \neq \varnothing$

1. We proved earlier that if X is finite, then $u(\cdot)$ is a utility function representing a rational $\succsim$. (by induction. Remember??)
2. Since X is finite, for any $B \subset X$ with $B \neq \varnothing$ there exists $x \in C(B)$ such that $u(x) \geq u(y)$ for all $y \in B$...
3. ...remember that finiteness implied that we could order all alternatives in X, and assign a value.
4. Then, it means that $x \in C^*(B, \succsim)$, i.e. x is chosen according to preference relation in B.
5. Hence, $C^*(B, \succsim)$ cannot be empty: $C^*(B, \succsim) \neq \varnothing$

□