Problem Set III: Comparative static, monotonicity, Cobb-Douglas

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1. MWG 2.F.10: substitution matrix

Consider the following demand function:

$$x_1(p,w) = \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1}$$
$$x_2(p,w) = \frac{p_3}{p_1+p_2+p_3}\frac{w}{p_2}$$
$$x_3(p,w) = \frac{p_1}{p_1+p_2+p_3}\frac{w}{p_3}$$

1. Compute the substitution matrix. Show that at $p = (1,1,1)$ and $w = 1$ it is negative semidefinite but not symmetric.
2. Show that this demand function does not satisfy the weak axiom. [Hint: consider $p = (1,1,\varepsilon)$ and show that the matrix is not negative semidefinite for $\varepsilon > 0$ small].

Recap: substitution matrix I

Definition 1 (Slutsky substitution matrix)**.** The *substitution matrix* $S(p,w)$ measures the differential change in the consumption of commodity l due to a differential change in the price of commodity k, when wealth is adjusted so that the consumer can still just afford his original consumption bundle. The general element of the matrix $S(p,w)$ has the form

$$s_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w}x_k(p,w)$$

The matrix in other words tells us the change in the consumption of commodity l solely due to a change in relative prices, when the wealth effect of the price change has been compensated.

Solution: computing the substitution matrix

- We will have to compute the derivative for all couples l,k, and then substitute $p = (1,1,1)$ and $w = 1$.
- As an example, the first element of the matrix is given by

$$s_{11}(p,w) = -\frac{p_2}{(p_1+p_2+p_3)^2}\frac{w}{p_1} - \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1^2} + \left(\frac{p_2}{(p_1+p_2+p_3)p_1}\right)\frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1}$$

- Substituting $p = (1,1,1)$ and $w = 1$ we get

$$s_{11}(p,w) = -\frac{1}{9} - \frac{1}{3} + \left(\frac{1}{3}\right)\frac{1}{3} = -\frac{1}{3}$$

- repeating the same calculation over all the matrix, we would get

$$s_{lk}(p,w) = \frac{1}{3}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Solution: negative semidefiniteness

- The matrix is obviously not symmetric.
- We must then study the definiteness of the matrix, without the usual criteria for symmetric matrices (eigenvalues, k_{th} order principal minors, north-west minors...).
- We will use two results:
  1. If $x(p,w)$ is homogeneous of degree zero and satisfies Walras' law, then $p \cdot S(p,w) = 0$ and $S(w,p) \cdot p = 0$
  2. If M is a square matrix that has the above property and $\hat{M}$ is the matrix obtained by deleting one row and the corresponding column, then M is negative semidefinite iff $\hat{M}$ is negative definite.
- we will hence have to prove that $\hat{M}$ is negative definite.
- This can be done by studying the sign of the quadratic form $z'\hat{M}z$:

$$[v_1, v_2] \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -v_1^2 + v_1 v_2 - v_2^2 = -\left(v_1 - \frac{v_2}{2}\right)^2 - \frac{3}{4}v_2^2$$

- which is always negative. Hence the minor studied is negative definite, and the matrix $S(w,p)$ is negative semidefinite.

Recap: W.A.R.P. in the context of Walrasian demand

Situation	**Choice**	**Implication**
(p,w)	$x(p,w)$	$C(B) = x(p,w)$
(p',w')	$x(p',w')$	$C(B) = x(p',w')$

- Given the above two situations, we will apply the logic of the W.A.R.P...
- ...which imposes consistency: if A is chosen when B is available, then in another choice when B is chosen, A must not be available.

Hypothesis	**Choice**	**Implication**
if $p \cdot x(p',w') \leq w$	$x(p,w)$	In situation (p,w), $x(p',w')$ is available, but *not* chosen
if $x(p',w') \neq x(p,w)$	-	The two demanded bundles are not identical
then $p' \cdot x(p,w) > w'$	$x(p',w')$	$x(p,w)$ must be unaffordable

- Since the agent reveals a preference for $x(p,w)$ when $x(p',w')$ is affordable, we require him to stick to this preference when both of them are affordable;
- if it does not, then it must mean (under the W.A.R.P.) that $x(p,w)$ was not affordable;
- or else, the W.A.R.P. is violated.

Solution: does this demand satisfy W.A.R.P.?

- We have a useful result: if a demand $x(p,w)$ satisfies Walras' law, homogeneity of degree zero and the W.A.R.P. at **any** (p,w), then the Slutsky $S(w,p)$ must be *negative semidefinite*.
- We checked that the matrix is negative semidefinite for $p = (1,1,1)$, but it is so for every (p,w)?
- We must hence check the definiteness of the matrix for a generic p.
- Let's consider the price of the third commodity as a variable, ε: $p = (1,1,\varepsilon)$.
- let's consider the 2×2 submatrix A obtained deleting last row and column:

$$A = \frac{1}{(2+\varepsilon)^2} \begin{bmatrix} -2-\varepsilon & 1+2\varepsilon \\ 0 & -3\varepsilon \end{bmatrix}$$

- If we choose the vector $z = [1, 3, 0]$ and we calculate the quadratic form for A, we get

$$z'Az = [1,3]\frac{1}{(2+\varepsilon)^2} \begin{bmatrix} -2-\varepsilon & 1+2\varepsilon \\ 0 & -3\varepsilon \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \cdots = \frac{1-20\varepsilon}{(2+\varepsilon)^2}$$

- which is positive for $\varepsilon > 0$ small (i.e. $\varepsilon < 0.05$).
- Hence $S(w, p)$ is not negative semidefinite for all price vectors, and W.A.R.P. is *not* satisfied.

2. MWG 2.F.17: demand

In an L-commodity world, a consumer's Walrasian demand function is

$$x_k(p, w) = \frac{w}{\sum_{l=1}^{L} p_l}, \text{for } k = 1, \ldots, L$$

1. Is this demand homogeneous of degree zero in (p, w)?
2. Does it satisfy Walras' Law?
3. Does it satisfy the Weak Axiom?
4. Compute the Slutsky substitution matrix for this demand function. Is it negative semidefinite? Negative definite? Symmetric?

Solution: points 1 and 2

Homogeneity of degree zero

- Let's just apply the definition: $x_k(\alpha p, \alpha w) = x_k(p, w)$

$$x_k(\alpha p, \alpha w) = \frac{\alpha w}{\sum_{l=1}^{L} \alpha p_l} = \frac{\alpha w}{\alpha \sum_{l=1}^{L} p_l} = \frac{w}{\sum_{l=1}^{L} p_l} = x_k(p, w)$$

Walras' law

- Again, let's apply the definition: $p \cdot x(p, w) = w$

$$p \cdot x(p, w) = \sum_{l=1}^{L} p_l x_l(p, w) = \sum_{l=1}^{L} p_l \frac{w}{\sum_{l=1}^{L} p_l} = w$$

Solution: point 3

- Suppose the situation is a potential violation of W.A.R.P.:
- $p' \cdot x(p, w) \leq w'$ and $p \cdot x(p', w') \leq w$;
- This means that both bundles are affordable every time (but choices differ).
- For the W.A.R.P. to hold, the bundles $x(p, w)$ and $x(p', w')$ must be *the same*

Proof.

1. By substitution, $p' \cdot x(p, w) \leq w'$ implies $\sum_l p'_l \frac{w}{\sum_l p_l} \leq w'$, i.e. $\frac{w}{\sum_l p_l} \leq \frac{w'}{\sum_l p'_l}$
2. and $p \cdot x(p', w') \leq w$ implies $\sum_l p_l \frac{w'}{\sum_l p'_l} \leq w$, i.e. $\frac{w'}{\sum_l p'_l} \leq \frac{w}{\sum_l p_l}$
3. Therefore it must be $\frac{w'}{\sum_l p'_l} = \frac{w}{\sum_l p_l}$;
4. Which means $x(p, w) = x(p', w')$

□

Solution: point 4

Slutsky substitution matrix

- Instead of computing element by element, we will compute the whole matrix in one go,
- since we will see it is more straightforward.
- The derivative of demand w.r.t. prices is given by the following $L \times L$ matrix:

$$D_p x(p,w) = -\frac{w}{(\sum_l p_l)^2} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}$$

- The compensated wealth effects are instead given by the column vector of pure wealth effects multiplied by the row vector obtained transposing the demand function $x(p,w)$ for all goods:

$$D_w x(p,w) \cdot x(p,w)^T = \frac{1}{\sum_l p_l} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \frac{w}{\sum_l p_l} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$$

...continued

- The two above expressions must be combined to get $S(w,p)$:

$$S(w,p) = -\frac{w}{(\sum_l p_l)^2} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} + \frac{w}{(\sum_l p_l)^2} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

- ...which, being the null matrix, is symmetric;
- and is negative semidefinite, even if not negative definite,
- since it has all eigenvalues equal to zero.

3. MWG 3.B.2: monotonicity

The preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ is said to be *weakly monotone* if and only if $x \geq y$ implies $x \succsim y$. Show that if $\succsim$ is transitive, locally non satiated and weakly monotone, then it is monotone.

Recap: monotonicity, local non-satiation

Definition 2 (Monotonicity). • The preference relation $\succsim$ on X is *monotone* if $x \in X$ and $y \gg x$ implies $y \succ x$.

- Intuitively, monotonicity implies that all goods are desirable: if a new bundle has more of every good, it is strictly preferred to the old bundle.

Definition 3 (Local non-satiation). • The preference relation $\succsim$ on X is *locally non-satiated* if for every $x \in X$ and every $\varepsilon > 0$ there is $y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.

- Intuitively, under local non-satiation it is enough to move a very small step from x and there will be a y that is preferred to x.
- Local non-satiation does not impose all goods to be desirable: y can also have less of some goods (but not of all, unless they are all 'bads').

Solution: Proof I

We have to prove that:

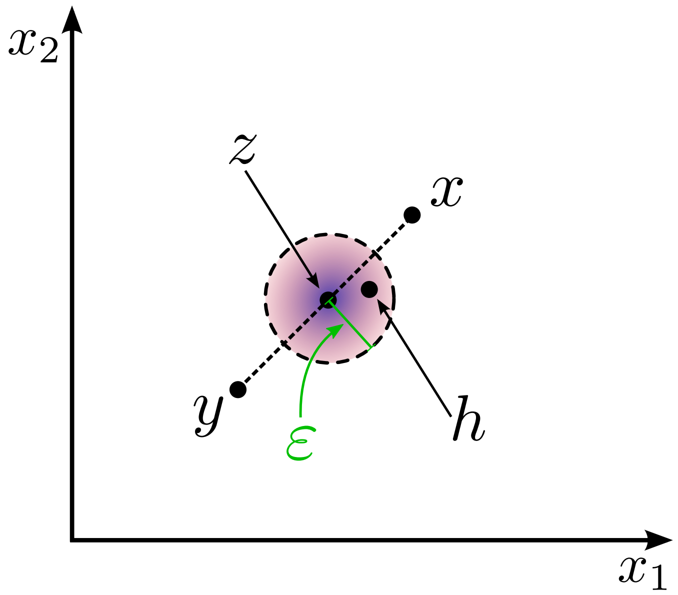
$$\succsim \quad \begin{array}{c}\text{transitive}\\ \text{loc.non-sat.}\\ \text{weakly monotone}\end{array} \quad \Rightarrow \quad \succsim \quad \text{monotone } (x \gg y \Rightarrow x \succ y)$$

- we will be using weak monotonicity and local non-satiation

Monotonicity.
- Let's take two bundles x, y with $x \gg y$ (i.e. $x_i > y_i \forall i$).
- By weak monotonicity, $x \succsim y$.
- Let's take a linear combination $z = \alpha x + (1 - \alpha)y,\ \alpha \in (0, 1)$
- We know that it must be $x \gg z \gg y$;
- by weak monotonicity, $x \succsim y \succsim z$ (see figure)

□

Solution: Graphics



Solution: Proof II

Monotonicity.
- Using local non satiation, we know that:
- $\forall \varepsilon > 0\ \exists h :\ \|z - h\| < \varepsilon$, such that $h \succ z$.
- ε can be chosen as small as we want, i.e. as small as to satisfy:
- $\varepsilon < \min\{\|x - z\|; \|z - y\|\}$.
- Choosing ε according to that condition ensures that $x \gg h \gg y$;
- By weak monotonicity, this means $x \succsim h \succsim y$.
- Summarising what we know:
- $x \succsim z \succsim y$, $x \succsim h \succsim y$ and $h \succ z$.
- putting it all together we must have
- $x \succsim h \succ z \succsim y$, which implies, by a theorem we proved in Problem Set 1 (have a look), $x \succ y$, Q.E.D.

□

Cobb-Douglas utility

Cobb-Douglas utility functions n $\mathbb{R}^2$, for two goods x, y are given by the general formula

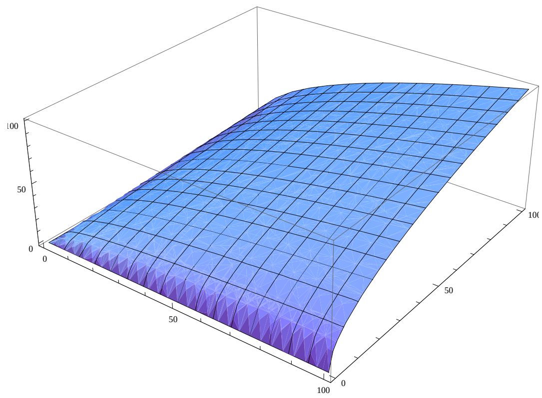
$$U = x^{\alpha} y^{1-\alpha}$$

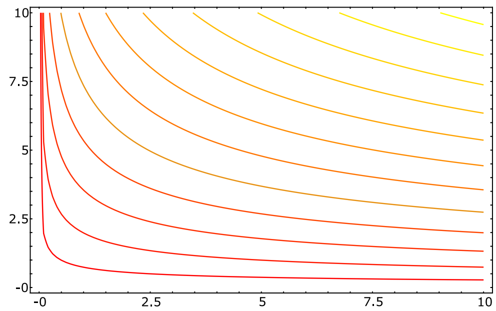
The function can be shown to be:

- monotonically increasing in the two arguments;
- concave;
- twice continuously differentiable with $U' > 0$ and $U'' < 0$;
- in other words, what economists call *well-behaved*.

We can have a look at both the 3D visualisation of the function and the level curves (indifference curves)

Cobb Douglas in 3D and indifference map in 2D





Solving a maximisation problem: steps

1. Gather ideas on the set of alternatives X;
2. Write down the budget constraint (and all other constraints);
3. Study the utility function; If more suitable, apply a strictly increasing (decreasing) transformation.
4. Write down the UMP as a Lagrangean (or as an unconstrained problem if you can);
5. Display and solve F.O.C.s;
6. The demand function is the set of values that maximises the utility for any given vector (p, w):
7. Hence, by plugging the values obtained with the F.O.C. in the budget constraint, and solving, we get the demand functions $x_l(p, w)$, $\forall l = 1 \ldots L$.

UMP with Cobb-Douglas utility I

Exercise
Let the utility function of a consumer be $U = x_1^{\alpha} x_2^{1-\alpha}$, and its budget constraint be $x_1 p_1 + x_2 p_2 \leq w$. Solve the Utility Maximisation Problem (UMP) and derive the demand functions $x_i(p, w)$, $i = 1, 2$ and the indirect utility function $v(p, w)$.

- Since U is increasing at all (p, w), the budget constraint will always be satisfied with equality;
- We can then formalise the UMP as

$$\max \; x_1^{\alpha} x_2^{1-\alpha}, \; s.t.: \; x_1 p_1 + x_2 p_2 = w$$

- The lagrangean can be setup to be:

$$\mathcal{L} = x_1^{\alpha} x_2^{1-\alpha} + \lambda (w - x_1 p_1 - x_2 p_2)$$

UMP with Cobb-Douglas utility II

- We can then compute the first order conditions (FOC), to be

$$\begin{cases} \dfrac{\partial U}{\partial x_1} = 0 \quad \Rightarrow \quad \alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1 \\ \dfrac{\partial U}{\partial x_2} = 0 \quad \Rightarrow \quad (1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda p_2 \end{cases}$$

- Dividing the first condition by the second we get

$$\frac{\alpha}{1-\alpha}\left(\frac{x_2}{x_1}\right) = \frac{p_1}{p_2}, \text{i.e.} \Rightarrow p_1 x_1 = \frac{\alpha}{1-\alpha} p_2 x_2$$

- And by combining this condition with the budget constraint, and solving by x_1 and x_2 in turn, we get

$$x_1(p,w) = \frac{\alpha w}{p_1} \quad x_2(p,w) = \frac{(1-\alpha)w}{p_2}$$

- which you can verify to be homogeneous of degree zero and satisfying Walras' law.

UMP: indirect utility function

- The indirect utility function $v(p,w)$ is the *value function* of the maximisation problem
- It is hence defined as $v(p,w) = u(x(p,w))$...
- ...i.e. by plugging the walrasian demand $x(p,w)$ into the utility function $u(x)$.
- It gives the value of the maximum attainable utility given prices and wealth.
- In this case, $v(p,w)$ is given by

$$v(p,w) = u(x(p,w)) = \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha}$$