Problem Set IV: UMP, EMP, indirect utility, expenditure

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Recap: indirect utility and marshallian demand

- The indirect utility function is the *value function* of the UMP:

$$v(p, w) = \max\ u(x) \text{ s.t. } p \cdot x \leq w$$

- Since the end result of the UMP are the Walrasian demand functions $x(p, w)$,
- the indirect utility function gives the optimal level of utility as a function of optimal demanded bundles,
- that is, ultimately, as a function of prices and wealth.

Summing up

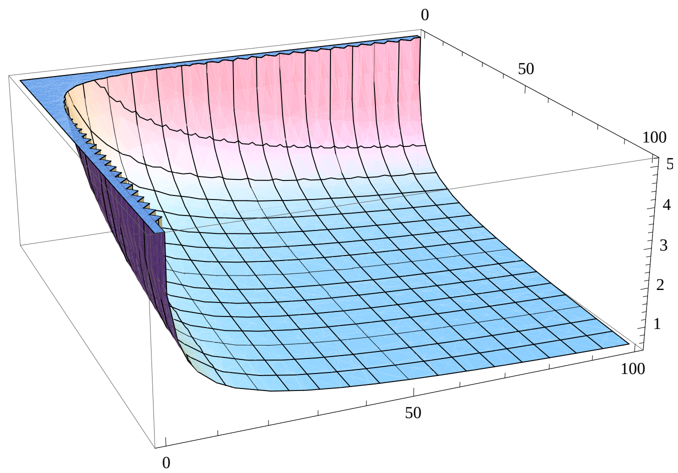
- In the UMP we assume a rational and locally non-satiated consumer with convex preferences that maximises utility;
- we hence find the optimally demanded bundles at any (p, w);
- The level of utility associated with any optimally demanded bundle is the indirect utility function $v(p, w)$.

Recap: properties of the indirect utility function

The value function of a standard UMP, the *indirect utility function* $v(p, w)$, is:

- Homogeneous of degree zero in p and w (doubling prices and wealth doesn't change anything);
- Strictly increasing in w and nonincreasing in p_l for any l (all income is spent; law of demand);
- Quasiconvex in p: that is, $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$ (see example in $\mathbb{R}^2$ in lecture slides);
- Continuous at all $p \gg 0$, $w > 0$ (from continuity of $u(x)$ and of $x(p, w)$).

Cobb-Douglas Indirect Utility Function, $\alpha = 0.5$**,** $w = 100$



Recap: expenditure function and hicksian demand

- The expenditure function is the *value function* of the EmP:

$$e(p,u) = \min\ p \cdot x \ \text{ s.t. } \ u(x) \geq u$$

- In the EmP we find the bundles that assure a fixed level of utility while minimizing expenditure
- the expenditure function gives the minimum level of expenditure needed to reach utility u when prices are p.

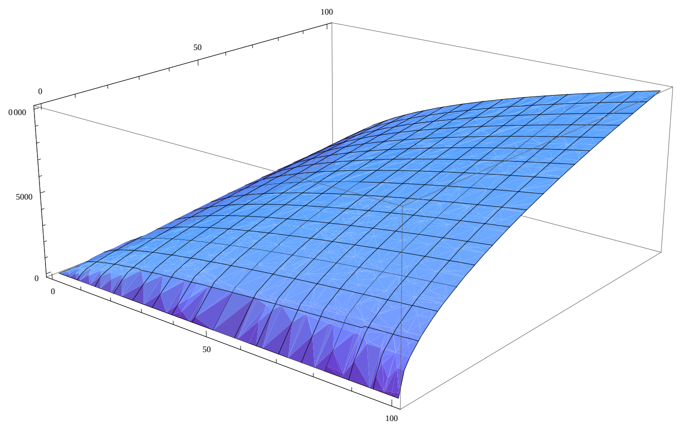
Summing up

- In the EmP we assume a rational and locally non-satiated consumer with convex preferences that minimises expenditure to reach a given level of utility;
- we denote the optimally demanded bundles at any (p,u) as $h(p,u)$ [hicksian demand];
- The level of expenditure associated with any optimally demanded bundle is the expenditure function $e(p,u)$.

Recap: properties of the expenditure function

- Homogeneous of degree one in p (expenditure is a linear function of prices);
- Strictly increasing in u and nondecreasing in p_l for any l (you spend more to achieve higher utility, you cannot spend less when prices go up);
- Concave in p (consumer adjusts to changes in prices doing at least not worse than linear change);
- Continuous in p and u (from continuity of $p \cdot x$ and $h(p,u)$).

Cobb-Douglas Expenditure Function, $\alpha = 0.5$**,** $u = 100$



Recap: basic duality relations

- The bundle that maximises utility is the same that minimises expenditure
- The indirect utility function gives the maximum utility obtainable with that bundle
- The wealth spent to obtain that utility is necessarily the minimum possible
- And spending all that wealth generates the maximum level of utility.

Four important identities

1. $v(p, e(p, u)) \equiv u$: the maximum level of utility attainable with minimal expenditure is u;
2. $e(p, v(p, w)) \equiv w$: the minimum expenditure necessary to reach optimal level of utility is w;
3. $x_i(p, w) \equiv h_i(p, v(p, w))$: the demanded bundle that maximises utility is the same as the demanded bundle that minimises expenditure at utility $v(p, w)$;
4. $h_i(p, u) \equiv x_i(p, e(p, u))$: the demanded bundle that minimises expenditure is the same as the demanded bundle that maximises utility at wealth $e(p, u)$.

Recap: a new look at the Slutsky matrix

- The hicksian demand $h(p, u)$ is also called the *compensated* demand.
- This reminds us of the Slutsky matrix, that gave us the *compensated* changes in demand for changes in prices.

$$\frac{\partial h(p,u)}{\partial p_k} = \frac{\partial x(p,w)}{\partial p_k} + \frac{\partial x(p,w)}{\partial w} \cdot x_k(p,w)$$

- In which the second term is exactly the lk entry of the Slutsky substitution matrix we are by now familiar with.
- This equation links the derivatives of the hicksian and walrasian demand functions:
- The two demands are the same when the wealth effect of a price change is compensated away.

Recap: Shephard's lemma

There are direct and straightforward relationships between $e(p, u)$ and $h(p, u)$.

1. $e(p, u)$ can be calculated by plugging the optimal demanded bundle under the EmP, $h(p, u)$, into the expression for calculating expenditure $p \cdot x$. Hence, $e(p, u) = p \cdot h(p, u)$.
2. Running in the opposite direction, it can be proved that $h(p, u) = \nabla_p e(p, u)$. This is mathematically the *Shephard's Lemma* (though the Lemma was derived from production theory, it is formally the same as the one exposed here).

Recap: Roy's identity

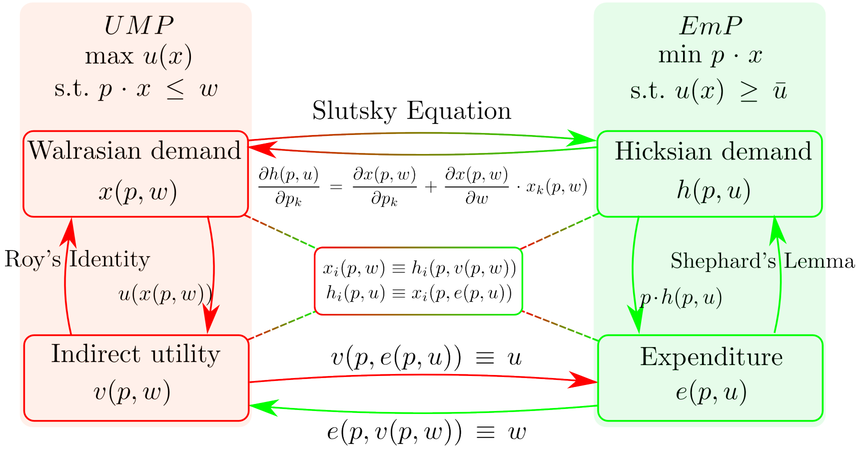
The relationships between $v(p, w)$ and $x(p, w)$ are less straightforward, but of the same kind:

1. $v(p, w)$ can be calculated by plugging the optimal demanded bundle under the UMP into the utility function, i.e. $v = u(x(p, w))$,
2. Going in the opposite direction is more tricky, since we are dealing with utility, an ordinal concept; in the case of expenditure we were dealing with a cardinal concept, money.

- In order to go from Walrasian demand to the Indirect Utility function we need to sterilise wealth effects and take into account the ordinality of the concepts;
- It can be proved that:

$$x_l(p,w) = -\frac{\dfrac{\partial v(p,w)}{\partial p_l}}{\dfrac{\partial v(p,w)}{\partial w}}$$

Recap: finding one's way through all of this



1. Varian 7.4: UMP-EMP

Consider the indirect utility function given by

$$v(p_1, p_2, w) = \frac{w}{p_1 + p_2}$$

1. What are the Walrasian demand functions?
2. What is the expenditure function?
3. What is the direct utility function?

Solution I

Walrasian demand functions
Walrasian demand functions can be derived from the indirect utility function using Roy's Identity:

$$x_l(p, w) = -\frac{\partial v(p, w)}{\partial p_l} \left(\frac{\partial v(p, w)}{\partial w} \right)^{-1}$$

In this case, plugging in the derivatives for the function,

$$x_1(p, w) = -\left(\frac{-w}{(p_1 + p_2)^2} \frac{p_1 + p_2}{1} \right) = \frac{w}{p_1 + p_2}$$

It can be verified that the same holds for $x_2(p, w)$. Hence the demand function is given by

$$x_1(p, w) = x_2(p, w) = \frac{w}{p_1 + p_2}$$

Solution II

Expenditure function
The expenditure function is the inverse of the indirect utility function with respect to wealth w:

$$v(p, e(p, u)) = u$$

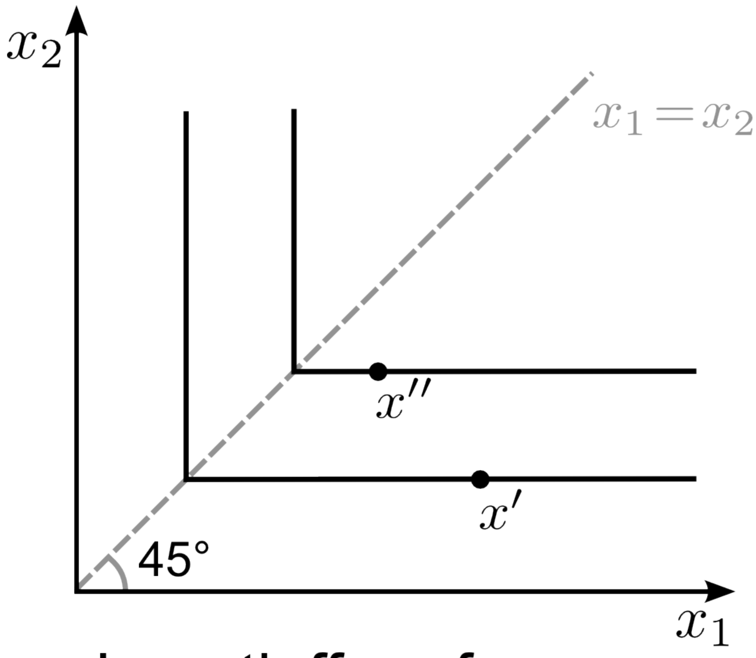
In this case, applying the above formula is enough to get the result:

$$\frac{e(p, u)}{p_1 + p_2} = u \;\Rightarrow\; e(p, u) = (p_1 + p_2)u$$

Solution III

Direct utility function
There is no easy automatic way to retrieve the utility function from indirect utility. We need to 'invert' a maximum process, which is not trivial, or else to work on the indirect utility and walrasian demand by 'inverting' the substitution.

- In this case, we see a striking regularity: the indirect utility funtion is the same as the demand functions.
- It means that the optimal level of utility is reached when only one of the two goods is consumed.
- It is then the case of perfect complement goods, i.e. Leontieff preferences.
- The resulting utility function is then $u(x) = \min\{x_1, x_2\}$



2. MWG 3.D.6: Stone linear expenditure system

Consider the following utility function in a three-good setting:

$$u(x) = (x_1 - b_1)^{\alpha}(x_2 - b_2)^{\beta}(x_3 - b_3)^{\gamma}$$

Assume that $\alpha + \beta + \gamma = 1$.

1. Write down the FOC for the UMP and derive the consumer's Walrasian demand and the indirect utility function.
2. Verify that the derived functions satisfy the following properties:
   (a) Walrasian demand $x(p, w)$ is homogeneous of degree zero and satisfies Walras' law;
   (b) Indirect utility $v(p, w)$ is homogeneous of degree zero;
   (c) $v(p, w)$ is strictly increasing in w and nonincreasing in p_l for all l;
   (d) $v(p, w)$ is continuous in p and w.

UMP I

We will work better with a log transform of the utility function:

$$\hat{u}(x) = \ln u(x) = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3)$$

which will give us the following UMP:

$$\max \ \hat{u}(x) \quad \text{s.t.} \ \ p \cdot x \leq w$$

Which, in turns, can be maximised using Lagrange method, to yield the following FOCs:

$$\frac{\alpha}{x_1 - b_1} = \lambda p_1 \ ; \ \frac{\beta}{x_2 - b_2} = \lambda p_2 \ ; \ \frac{\gamma}{x_3 - b_3} = \lambda p_3 \ , \ p \cdot x = w \ ; \ \lambda > 0$$

UMP II

Demand

The system can be solved to find the walrasian demand function:

$$x(p,w) = \begin{bmatrix} b_1 + \dfrac{\alpha(w - p \cdot b)}{p_1} \\ b_2 + \dfrac{\beta(w - p \cdot b)}{p_2} \\ b_3 + \dfrac{\gamma(w - p \cdot b)}{p_3} \end{bmatrix} \text{, in which } p \cdot b = \sum_{i=1}^{3} p_i b_i$$

Indirect utility

Given this demand funtion, the indirect utility can be found by substitution:

$$v(p,w) = u(x(p,w)) = \left(\frac{\alpha(w - p \cdot b)}{p_1}\right)^{\alpha} \left(\frac{\beta(w - p \cdot b)}{p_2}\right)^{\beta} \left(\frac{\gamma(w - p \cdot b)}{p_3}\right)^{\gamma}$$

Properties of $x(p,w)$

Homogeneity of degree zero

$$x(\lambda p, \lambda w) = \begin{bmatrix} b_1 + \dfrac{\alpha\lambda(w - p \cdot b)}{\lambda p_1} \\ b_2 + \dfrac{\beta\lambda(w - p \cdot b)}{\lambda p_2} \\ b_3 + \dfrac{\gamma\lambda(w - p \cdot b)}{\lambda p_3} \end{bmatrix} = x(p,w)$$

Walras' law

$$p \cdot x(p,w) = p \cdot b + (w - p \cdot b)\left(p_1 \frac{\alpha}{p_1} + p_2 \frac{\beta}{p_2} + p_3 \frac{\gamma}{p_3}\right) =$$
$$= p \cdot b + (w - p \cdot b)(\alpha + \beta + \gamma) = p \cdot b + w - p \cdot b = w$$

Properties of $v(p,w)$ I

Homogeneity of indirect utility

$$v(\lambda p, \lambda w) = \left(\frac{\alpha\lambda(w - p \cdot b)}{\lambda p_1}\right)^{\alpha} \left(\frac{\beta\lambda(w - p \cdot b)}{\lambda p_2}\right)^{\beta} \left(\frac{\gamma\lambda(w - p \cdot b)}{\lambda p_3}\right)^{\gamma}$$

which can easily be simplified to yield $v(p,w)$.

Derivatives

$v(p,w)$ strictly increasing in w: first simplify the indirect utility function to get

$$v(\lambda p, \lambda w) = (w - p \cdot b) \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{\beta}{p_2}\right)^{\beta} \left(\frac{\gamma}{p_3}\right)^{\gamma}$$

and then simply differentiate w.r.t. w to get

$$\frac{\partial v(p,w)}{\partial w} = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{\beta}{p_2}\right)^{\beta} \left(\frac{\gamma}{p_3}\right)^{\gamma} > 0$$

Properties of $v(p,w)$ II

Derivatives, continued
The derivatives w.r.t. prices imply long calculations, and yield:

$$\frac{\partial v}{\partial p_1} = v(p,w)\left(-\frac{\alpha}{p_1}\right) \quad \frac{\partial v}{\partial p_2} = v(p,w)\left(-\frac{\beta}{p_2}\right) \quad \frac{\partial v}{\partial p_3} = v(p,w)\left(-\frac{\gamma}{p_3}\right)$$

That can be checked to be all < 0, as required.

continuity
Continuity comes directly from the functional form: with $p \gg 0$, as assumed, there are no asympthotes or kinks. Moreover, the utility function and the derived walrasian demand being continuous, the indirect utility function has to be continuous.

3. MWG 3.G.15: dual properties

Consider the utility function

$$u = 2x_1^{\frac{1}{2}} + 4x_2^{\frac{1}{2}}$$

1. Find the demand functions $x_1(p,w)$ and $x_2(p,w)$
2. Find the compensated demand function $h(p,u)$
3. Find the expenditure function $e(p,u)$ and verify that $h(p,u) = \nabla_p e(p,u)$
4. Find the indirect utility function $v(p,w)$ and verfy Roy's identity.

Walrasian demand

Solution strategy
To find walrasian demand, just solve the UMP using Lagrange method.

The FOC system for this problem boils down to

$$\frac{1}{2}\left(\frac{x_1}{x_2}\right)^{-\frac{1}{2}} = \frac{p_1}{p_2} \;\; ; \;\; p_1x_1 + p_2x_2 = w$$

Yielding solution

$$x(p,w) = \begin{bmatrix} \dfrac{p_2 w}{4p_1^2 + p_1 p_2} \\ \dfrac{4p_1 w}{p_2^2 + 4p_1 p_2} \end{bmatrix}$$

Hicksian demand

Solution strategy
We need to find Hicksian demand, knowing $u(x)$ and $x(p,w)$. This can be done in two ways:

1. Using Slutsky equation we can find the derivative w.r.t. p of the Hicksian demand knowing $x(p,w)$. This is rather straightforward, but implies integrating. The steps are:
   - Compute derivatives of $x_l(p,w)$ w.r.t. p_l and w;
   - Apply the Slutsky equation to find $\frac{\partial h(p,w)}{\partial p_l}$;
   - Integrate $\int \frac{\partial h(p,u)}{\partial p_l} dp_l$ to get $h_l(p,w)$.
2. Exploting the identity $h(p,w) \equiv x(p,e(p,u)$. This eliminates the need for integration, but implies calculating the indirect utility function and from there the expenditure function. The steps are:
   - Plug the demand functions into $u(x)$ to get $v(p,w)$;
   - Apply $v(p,e(p,u)) = u$, i.e. invert v w.r.t. wealth w;
   - Apply $h(p,w) \equiv x(p,e(p,u))$, i.e. substitute w with $e(p,u)$ in the Walrasian demand.

We will follow road 2. This means answering further questions first

The road to Hicksian demand I

- Plug the demand functions into $u(x)$ to get $v(p,w)$:

$$v(p,w) = u(x(p,w)) = 2\left(\frac{p_2 w}{4p_1^2 + p_1 p_2}\right)^{\frac{1}{2}} + 4\left(\frac{4p_1 w}{p_2^2 + 4p_1 p_2}\right)^{\frac{1}{2}}$$

- Apply $v(p,e(p,u)) = u$, i.e. invert v w.r.t. wealth w;

$$2\left(\frac{p_2 e(p,u)}{4p_1^2 + p_1 p_2}\right)^{\frac{1}{2}} + 4\left(\frac{4p_1 e(p,u)}{p_2^2 + 4p_1 p_2}\right)^{\frac{1}{2}} = u$$

which, squaring both sides and then simplifying, gives two roots, one of which is negative, the one remaining being:

$$e(p,u) = \frac{1}{4}\frac{u^2 p_1 p_2}{4p_1 + p_2}$$

The road to Hicksian demand II

- We are left with the last step, i.e. applying $h(p,w) \equiv x(p,e(p,u))$:
- i.e. we have to substitute the $e(p,u)$ we found in the place of w.

$$h_1(p,u) = \frac{1}{4}\frac{p_1 p_2^2 u^2}{(p_2 + 4p_1)(p_1 p_2 + 4p_1^2)} \;\; ; \;\; h_2(p,u) = \frac{p_1^2 p_2 u^2}{(p_2 + 4p_1)(4p_1 p_2 + p_2^2)}$$

That can be simplyfied to yield

$$h_1(p,u) = \frac{1}{4}\left(\frac{p_2 u}{4p_1 + p_2}\right)^2 \;\; ; \;\; h_2(p,u) = \left(\frac{p_1 u}{4p_1 + p_2}\right)^2$$

Expenditure function

Solution strategy
Again, we have two ways of finding the expenditure function:

1. Retrieve $v(p,w)$ from $x(p,w)$ and $u(x)$, then invert it w.r.t. w to get $e(p,u)$;
2. Retrieve $e(p,u)$ directly from $h(p,u)$ plugging it in the objective function $p \cdot x$.

- As for us, we used road 1 and already worked out $e(p,u)$ in the road towards Hicksian demand, so no need to do it here.
- You can easily check by yourself that $h(p,u) = \nabla_p e(p,u)$

Roy's identity

Solution strategy
We can find $v(p,w)$ fromn either $v(p,w) = u(x(p,w))$ or inverting $e(p,u)$ w.r.t. u; then, we just need to apply Roy's identity right hand side and check if the result is the same as the $x(p,w)$ we calculated beforehand. We have to check if this holds:

$$x_l(p,w) = -\frac{\partial v(p,w)}{\partial p_l}\left(\frac{\partial v(p,w)}{\partial w}\right)^{-1}, \text{ for } l = 1,2$$

As for us, we already found $v(p,w)$. It's easy again to apply the formula and find that Roy's Identity holds