Problem Set IV: UMP, EMP, indirect utility, expenditure

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Recap: indirect utility and marshallian demand

- The indirect utility function is the *value function* of the UMP:
  \[ v(p, w) = \max u(x) \text{ s.t. } p \cdot x \leq w \]

- Since the end result of the UMP are the Walrasian demand functions \( x(p, w) \),
- the indirect utility function gives the optimal level of utility as a function of optimal demanded bundles,
- that is, ultimately, as a function of prices and wealth.

Summing up

- In the UMP we assume a rational and locally non-satiated consumer with convex preferences that maximises utility;
- we hence find the optimally demanded bundles at any \((p, w)\);
- The level of utility associated with any optimally demanded bundle is the indirect utility function \( v(p, w) \).
Recap: properties of the indirect utility function

The value function of a standard UMP, the *indirect utility function* \( v(p, w) \), is:

- Homogeneous of degree zero in \( p \) and \( w \) (doubling prices and wealth doesn’t change anything);
- Strictly increasing in \( w \) and nonincreasing in \( p_l \) for any \( l \) (all income is spent; law of demand);
- Quasiconvex in \( p \): that is, \( \{ (p, w) : v(p, w) \leq \bar{v} \} \) is convex for any \( \bar{v} \) (see example in \( \mathbb{R}^2 \) in lecture slides);
- Continuous at all \( p \gg 0, w > 0 \) (from continuity of \( u(x) \) and of \( x(p, w) \)).
Cobb-Douglas Indirect Utility Function, $\alpha = 0.5, \omega = 100$
Recap: expenditure function and hicksian demand

• The expenditure function is the value function of the EmP:
  \[ e(p, u) = \min p \cdot x \text{ s.t. } u(x) \geq u \]

• In the EmP we find the bundles that assure a fixed level of utility while minimizing expenditure.
• The expenditure function gives the minimum level of expenditure needed to reach utility \( u \) when prices are \( p \).

Summing up

• In the EmP we assume a rational and locally non-satiated consumer with convex preferences that minimises expenditure to reach a given level of utility;
• We denote the optimally demanded bundles at any \((p, u)\) as \( h(p, u) \) [hicksian demand];
• The level of expenditure associated with any optimally demanded bundle is the expenditure function \( e(p, u) \).
Recap: properties of the expenditure function

- Homogeneous of degree one in \( p \) (expenditure is a linear function of prices);
- Strictly increasing in \( u \) and nondecreasing in \( p_l \) for any \( l \) (you spend more to achieve higher utility, you cannot spend less when prices go up);
- Concave in \( p \) (consumer adjusts to changes in prices doing at least not worse than linear change);
- Continuous in \( p \) and \( u \) (from continuity of \( p \cdot x \) and \( h(p, u) \)).
Cobb-Douglas Expenditure Function, $\alpha = 0.5, u = 100$
Recap: basic duality relations

- The bundle that maximises utility is the same that minimises expenditure
- The indirect utility function gives the maximum utility obtainable with that bundle
- The wealth spent to obtain that utility is necessarily the minimum possible
- And spending all that wealth generates the maximum level of utility.

Four important identities

1. $v(p, e(p, u)) \equiv u$: the maximum level of utility attainable with minimal expenditure is $u$;
2. $e(p, v(p, w)) \equiv w$: the minimum expenditure necessary to reach optimal level of utility is $w$;
3. $x_i(p, w) \equiv h_i(p, v(p, w))$: the demanded bundle that maximises utility is the same as the demanded bundle that minimises expenditure at utility $v(p, w)$;
4. $h_i(p, u) \equiv x_i(p, e(p, u))$: the demanded bundle that minimises expenditure is the same as the demanded bundle that maximises utility at wealth $e(p, u)$.
Recap: a new look at the Slutsky matrix

- The hicksian demand \( h(p, u) \) is also called the \textit{compensated} demand.
- This reminds us of the Slutsky matrix, that gave us the \textit{compensated} changes in demand for changes in prices.

\[
\frac{\partial h(p, u)}{\partial p_k} = \frac{\partial x(p, w)}{\partial p_k} + \frac{\partial x(p, w)}{\partial w} \cdot x_k(p, w)
\]

- In which the second term is exactly the \( lk \) entry of the Slutsky substitution matrix we are by now familiar with.
- This equation links the derivatives of the hicksian and walrasian demand functions:
- The two demands are the same when the wealth effect of a price change is compensated away.
Recap: Shephard’s lemma

There are direct and straightforward relationships between $e(p, u)$ and $h(p, u)$.

1. $e(p, u)$ can be calculated by plugging the optimal demanded bundle under the EmP, $h(p, u)$, into the expression for calculating expenditure $p \cdot x$. Hence, $e(p, u) = p \cdot h(p, u)$.

2. Running in the opposite direction, it can be proved that $h(p, u) = \nabla_p e(p, u)$. This is mathematically the Shephard’s Lemma (though the Lemma was derived from production theory, it is formally the same as the one exposed here).
Recap: Roy’s identity

The relationships between \( v(p, w) \) and \( x(p, w) \) are less straightforward, but of the same kind:

1. \( v(p, w) \) can be calculated by plugging the optimal demanded bundle under the UMP into the utility function, i.e. \( v = u(x(p, w)) \),

2. Going in the opposite direction is more tricky, since we are dealing with utility, an ordinal concept; in the case of expenditure we were dealing with a cardinal concept, money.

- In order to go from Walrasian demand to the Indirect Utility function we need to sterilise wealth effects and take into account the ordinality of the concepts;
- It can be proved that:

\[
x_l(p, w) = - \frac{\partial v(p, w)}{\partial w} \frac{\partial p_l}{\partial v(p, w)}
\]
Recap: finding one’s way through all of this

\[ UMP \]
\[
\text{max } u(x) \\
\text{s.t. } p \cdot x \leq w
\]

Walrasian demand
\[ x(p, w) \]

Roy’s Identity
\[ u(x(p, w)) \]

Indirect utility
\[ v(p, w) \]

\[ EmP \]
\[
\text{min } p \cdot x \\
\text{s.t. } u(x) \geq \bar{u}
\]

Hicksian demand
\[ h(p, u) \]

Shephard’s Lemma
\[ p \cdot h(p, u) \]

Expenditure
\[ e(p, v(p, w)) \equiv w \]

\[ v(p, e(p, u)) \equiv u \]

Slutsky Equation
\[
\frac{\partial h(p, u)}{\partial p_k} = \frac{\partial x(p, w)}{\partial p_k} + \frac{\partial x(p, w)}{\partial w} \cdot x_k(p, w)
\]
Consider the indirect utility function given by

\[ v(p_1, p_2, w) = \frac{w}{p_1 + p_2} \]

1. What are the Walrasian demand functions?
2. What is the expenditure function?
3. What is the direct utility function?
Solution I

Walrasian demand functions
Walrasian demand functions can be derived from the indirect utility function using Roy’s Identity:

\[ x_i(p, w) = -\frac{\partial v(p, w)}{\partial p_i} \left( \frac{\partial v(p, w)}{\partial w} \right)^{-1} \]

In this case, plugging in the derivatives for the function,

\[ x_1(p, w) = -\left( \frac{-w}{(p_1 + p_2)^2} \frac{p_1 + p_2}{1} \right) = \frac{w}{p_1 + p_2} \]

It can be verified that the same holds for \( x_2(p, w) \). Hence the demand function is given by

\[ x_1(p, w) = x_2(p, w) = \frac{w}{p_1 + p_2} \]
Solution II

Expenditure function
The expenditure function is the inverse of the indirect utility function with respect to wealth \( w \):

\[ v(p, e(p, u)) = u \]

In this case, applying the above formula is enough to get the result:

\[ \frac{e(p, u)}{p_1 + p_2} = u \Rightarrow e(p, u) = (p_1 + p_2)u \]
Solution III

Direct utility function
There is no easy automatic way to retrieve the utility function from indirect utility. We need to ’invert’ a maximum process, which is not trivial, or else to work on the indirect utility and walrasian demand by ’inverting’ the substitution.

• In this case, we see a striking regularity: the indirect utility function is the same as the demand functions.
• It means that the optimal level of utility is reached when only one of the two goods is consumed.
• It is then the case of perfect complement goods, i.e. Leontieff preferences.
• The resulting utility function is then \( u(x) = \min\{x_1, x_2\} \)
Leontieff preferences
2. MWG 3.D.6: Stone linear expenditure system

Consider the following utility function in a three-good setting:

\[ u(x) = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma \]

Assume that \( \alpha + \beta + \gamma = 1 \).

1. Write down the FOC for the UMP and derive the consumer’s Walrasian demand and the indirect utility function.

2. Verify that the derived functions satisfy the following properties:
   2.1 Walrasian demand \( x(p, w) \) is homogeneous of degree zero and satisfies Walras’ law;
   2.2 Indirect utility \( v(p, w) \) is homogeneous of degree zero;
   2.3 \( v(p, w) \) is strictly increasing in \( w \) and nonincreasing in \( p_l \) for all \( l \);
   2.4 \( v(p, w) \) is continuous in \( p \) and \( w \).
We will work better with a log transform of the utility function:

\[ \hat{u}(x) = \ln u(x) = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3) \]

which will give us the following UMP:

\[ \max \hat{u}(x) \quad \text{s.t.} \quad p \cdot x \leq w \]

Which, in turns, can be maximised using Lagrange method, to yield the following FOCs:

\[ \frac{\alpha}{x_1 - b_1} = \lambda p_1; \quad \frac{\beta}{x_2 - b_2} = \lambda p_2; \quad \frac{\gamma}{x_3 - b_3} = \lambda p_3, \quad p \cdot x = w; \quad \lambda > 0 \]
UMP II

Demand
The system can be solved to find the walrasian demand function:

\[
x(p, w) = \begin{bmatrix}
    b_1 + \frac{\alpha(w - p \cdot b)}{p_1} \\
    b_2 + \frac{\beta(w - p \cdot b)}{p_2} \\
    b_3 + \frac{\gamma(w - p \cdot b)}{p_3}
\end{bmatrix}, \text{ in which } p \cdot b = \sum_{i=1}^{3} p_i b_i
\]

Indirect utility
Given this demand function, the indirect utility can be found by substitution:

\[
v(p, w) = u(x(p, w)) = \left(\frac{\alpha(w - p \cdot b)}{p_1}\right)^\alpha \left(\frac{\beta(w - p \cdot b)}{p_2}\right)^\beta \left(\frac{\gamma(w - p \cdot b)}{p_3}\right)^\gamma
\]
Properties of $x(p, w)$

Homogeneity of degree zero

$$x(\lambda p, \lambda w) = \left[ \begin{array}{c} b_1 + \frac{\alpha \lambda (w - p \cdot b)}{\lambda p_1} \\ b_2 + \frac{\beta \lambda (w - p \cdot b)}{\lambda p_2} \\ b_3 + \frac{\gamma \lambda (w - p \cdot b)}{\lambda p_3} \end{array} \right] = x(p, w)$$

Walras’ law

$$p \cdot x(p, w) = p \cdot b + (w - p \cdot b) \left( p_1 \frac{\alpha}{p_1} + p_2 \frac{\beta}{p_2} + p_3 \frac{\gamma}{p_3} \right) =$$

$$= p \cdot b + (w - p \cdot b)(\alpha + \beta + \gamma) = p \cdot b + w - p \cdot b = w$$
Properties of $\nu(p, w)$

Homogeneity of indirect utility

$$\nu(\lambda p, \lambda w) = \left( \frac{\alpha \lambda (w - p \cdot b)}{\lambda p_1} \right)^\alpha \left( \frac{\beta \lambda (w - p \cdot b)}{\lambda p_2} \right)^\beta \left( \frac{\gamma \lambda (w - p \cdot b)}{\lambda p_3} \right)^\gamma$$

which can easily be simplified to yield $\nu(p, w)$.

Derivatives

$\nu(p, w)$ strictly increasing in $w$: first simplify the indirect utility function to get

$$\nu(\lambda p, \lambda w) = (w - p \cdot b) \left( \frac{\alpha}{p_1} \right)^\alpha \left( \frac{\beta}{p_2} \right)^\beta \left( \frac{\gamma}{p_3} \right)^\gamma$$

and then simply differentiate w.r.t. $w$ to get

$$\frac{\partial \nu(p, w)}{\partial w} = \left( \frac{\alpha}{p_1} \right)^\alpha \left( \frac{\beta}{p_2} \right)^\beta \left( \frac{\gamma}{p_3} \right)^\gamma > 0$$
Properties of $\nu(p, w)$ II

Derivatives, continued
The derivatives w.r.t. prices imply long calculations, and yield:

$$\frac{\partial \nu}{\partial p_1} = \nu(p, w) \left( -\frac{\alpha}{p_1} \right) \quad \frac{\partial \nu}{\partial p_2} = \nu(p, w) \left( -\frac{\beta}{p_2} \right) \quad \frac{\partial \nu}{\partial p_3} = \nu(p, w) \left( -\frac{\gamma}{p_3} \right)$$

That can be checked to be all $< 0$, as required.

continuity
Continuity comes directly from the functional form: with $p \gg 0$, as assumed, there are no asymptotes or kinks. Moreover, the utility function and the derived walrasian demand being continuous, the indirect utility function has to be continuous.
3. MWG 3.G.15: dual properties

Consider the utility function

\[ u = 2x_1^{\frac{1}{2}} + 4x_2^{\frac{1}{2}} \]

1. Find the demand functions \( x_1(p, w) \) and \( x_2(p, w) \)
2. Find the compensated demand function \( h(p, u) \)
3. Find the expenditure function \( e(p, u) \) and verify that \( h(p, u) = \nabla_p e(p, u) \)
4. Find the indirect utility function \( v(p, w) \) and verify Roy’s identity.
Walrasian demand

Solution strategy
To find walrasian demand, just solve the UMP using Lagrange method.
The FOC system for this problem boils down to

\[
\frac{1}{2} \left( \frac{x_1}{x_2} \right)^{-\frac{1}{2}} = \frac{p_1}{p_2} ; \quad p_1 x_1 + p_2 x_2 = w
\]

Yielding solution

\[
x(p, w) = \begin{bmatrix}
p_2 w \\
4p_1^2 + p_1 p_2 \\
4p_1 w \\
p_2^2 + 4p_1 p_2
\end{bmatrix}
\]
Hicksian demand

Solution strategy

We need to find Hicksian demand, knowing \( u(x) \) and \( x(p, w) \). This can be done in two ways:

1. Using Slutsky equation we can find the derivative w.r.t. \( p \) of the Hicksian demand knowing \( x(p, w) \). This is rather straightforward, but implies integrating. The steps are:
   - Compute derivatives of \( x_l(p, w) \) w.r.t. \( p_l \) and \( w \);
   - Apply the Slutsky equation to find \( \frac{\partial h(p, w)}{\partial p_l} \);
   - Integrate \( \int \frac{\partial h(p, u)}{\partial p_l} dp_l \) to get \( h_l(p, w) \).

2. Exploiting the identity \( h(p, w) \equiv x(p, e(p, u)) \). This eliminates the need for integration, but implies calculating the indirect utility function and from there the expenditure function. The steps are:
   - Plug the demand functions into \( u(x) \) to get \( v(p, w) \);
   - Apply \( v(p, e(p, u)) = u \), i.e. invert \( v \) w.r.t. wealth \( w \);
   - Apply \( h(p, w) \equiv x(p, e(p, u)) \), i.e. substitute \( w \) with \( e(p, u) \) in the Walrasian demand.

We will follow road 2. This means answering further questions first
The road to Hicksian demand

• Plug the demand functions into \( u(x) \) to get \( v(p, w) \):

\[
v(p, w) = u(x(p, w)) = 2 \left( \frac{p_2 w}{4p_1^2 + p_1 p_2} \right)^{\frac{1}{2}} + 4 \left( \frac{4p_1 w}{p_2^2 + 4p_1 p_2} \right)^{\frac{1}{2}}
\]

• Apply \( v(p, e(p, u)) = u \), i.e. invert \( v \) w.r.t. wealth \( w \);

\[
2 \left( \frac{p_2 e(p, u)}{4p_1^2 + p_1 p_2} \right)^{\frac{1}{2}} + 4 \left( \frac{4p_1 e(p, u)}{p_2^2 + 4p_1 p_2} \right)^{\frac{1}{2}} = u
\]

which, squaring both sides and then simplifying, gives two roots, one of which is negative, the one remaining being:

\[
e(p, u) = \frac{1}{4} \frac{u^2 p_1 p_2}{4p_1 + p_2}
\]
The road to Hicksian demand II

- We are left with the last step, i.e. applying \( h(p, w) \equiv x(p, e(p, u)) \):
- i.e. we have to substitute the \( e(p, u) \) we found in the place of \( w \).

\[
\begin{align*}
h_1(p, u) &= \frac{1}{4} \frac{p_1p_2^2u^2}{(p_2 + 4p_1)(p_1p_2 + 4p_1^2)}; \\
h_2(p, u) &= \frac{p_1^2p_2u^2}{(p_2 + 4p_1)(4p_1p_2 + p_2^2)}
\end{align*}
\]

That can be simplified to yield

\[
\begin{align*}
h_1(p, u) &= \frac{1}{4} \left( \frac{p_2u}{4p_1 + p_2} \right)^2; \\
h_2(p, u) &= \left( \frac{p_1u}{4p_1 + p_2} \right)^2
\end{align*}
\]
Expenditure function

Solution strategy
Again, we have two ways of finding the expenditure function:

1. Retrieve $v(p, w)$ from $x(p, w)$ and $u(x)$, then invert it w.r.t. $w$ to get $e(p, u)$;
2. Retrieve $e(p, u)$ directly from $h(p, u)$ plugging it in the objective function $p \cdot x$.

• As for us, we used road 1 and already worked out $e(p, u)$ in the road towards Hicksian demand, so no need to do it here.
• You can easily check by yourself that $h(p, u) = \nabla_p e(p, u)$
Roy’s identity

Solution strategy

We can find \( v(p, w) \) from either \( v(p, w) = u(x(p, w)) \) or inverting \( e(p, u) \) w.r.t. \( u \); then, we just need to apply Roy’s identity right hand side and check if the result is the same as the \( x(p, w) \) we calculated beforehand. We have to check if this holds:

\[
x_l(p, w) = -\frac{\partial v(p, w)}{\partial p_l} \left( \frac{\partial v(p, w)}{\partial w} \right)^{-1}, \text{ for } l = 1, 2
\]

As for us, we already found \( v(p, w) \). It’s easy again to apply the formula and find that Roy’s Identity holds