Problem Set V: production

Paolo Crosetto
paolo.crosetto@unimi.it

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Recap: notation, production set Y, netput vectors

- A production vector, or *netput* vector, is denoted by $y = (y_1, \ldots, y_L) \in \mathbb{R}^L$;
- negative entries represent inputs, positive entries output.
- The set of all technologically feasible plans is $Y \subset \mathbb{R}^L$.

- In the case of a single-output, many input technology, we use a different notation:
- In this case y has $L-1$ nonpositive entries (inputs) and 1 nonnegative entry (output).
- We denote inputs by a *positive* vector $z = (z_1, \ldots, z_{L-1}) \in \mathbb{R}_+^{L-1}$, and output with the scalar $q \geq 0$.
- In this case a production function tells us how much output q is technologically possible to produce given z inputs:
- all feasible plans will hence have the property $q \leq f(z)$.

Recap: properties of Y

No free lunch: no input must imply no output. No creation *ex-nihilo*.

Inaction allowed: not doing anything is possible [sunk cost]

Free disposal: it is always possible to get rid for free of additional inputs.

Irreversibility: it is not possible to reverse a technology. [thermodinamics]

Additivity: if two production plans are feasible, producing both plans is also feasible. [entry]

CRS: scaling production up by α increases production by α.

IRS: scaling production up by α increases production by *more than* α.

DRS: scaling production up by α increases production by *less than* α.

Convexity: A convex combinations of two vectors $\in Y$ is also in Y.

Single output: profit maximisation

In all of the following, p is price of output q and $w = (w_1, \ldots, w_{L-1})$ is price of inputs z.

The profit maximisation problem is solved by maximising revenues minus cost:

$$\Pi(p, w) = \max_{z \geq 0} \; pf(z) - w \cdot z$$

- The result of the maximisation problem, $z(p, w)$, is the optimal amount of inputs required to maximise profit. It is the demand function for inputs.
- The amount produced at $z(p, w)$, $y(p, w) = f(z(p, w))$, is the firm's supply function (correspondence);
- Given $z(p, w)$, the profit function $\Pi(p, w)$ can be found by plugging $z(p, w)$ in the maximand;
- Given $\Pi(p, w)$, *Hotelling's Lemma* tells us that $y(p, w) = \nabla_p \Pi(p, w)$.

Single output: cost minimisation

Since there is no budget constraint, here the two *dual* problems are more tied together than in consumer theory.

The cost minimisation problem is solved by calculating the minimal amount of input needed to reach production level q:

$$c(w, q) = \min_{z \geq 0} \; w \cdot z \;\text{ s.t. } f(z) \geq q$$

- The result of the minimisation problem, $z(w, q)$, is the optimal amount of inputs required to minimise cost, and is called the *conditional demand function*.
- *Conditional* because it is the demand given production level q.
- Given $z(w, q)$, the cost function $c(w, q)$ can be found by plugging $z(w, q)$;
- Given $c(w, q)$, *Shepard's Lemma* tells us that $z(p, q) = \nabla_p c(w, q)$.

1. MWG 5.B.2: homogeneity

Let $f(\cdot)$ be the production function associated with a single-output technology, and let Y be the production set. Show that Y satisfies constant returns to scale if and only if $f(\cdot)$ is homogeneous of degree one.

Definition

Homogeneity of degree one A function $f(x)$ is homogeneous of degree one if $f(\alpha x) = \alpha f(x)$. Note that linear functions are homogeneous of degree one.

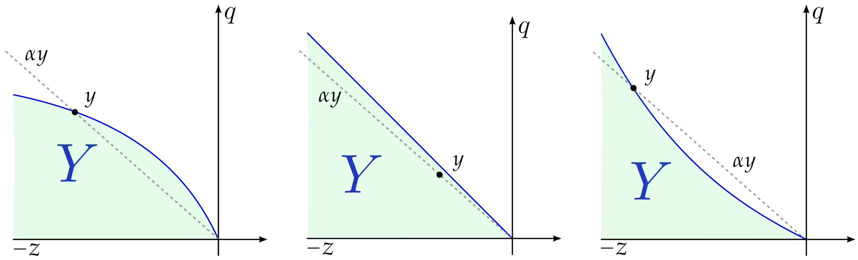
Definition

Constant returns to scale The production set Y exhibits constant returns to scale (CRS) if $y \in Y$ implies $\alpha y \in Y$ for any scalar $\alpha \geq 0$. That is, in a single input - single output technology, if $(-z, q) \in Y$ then $(-\alpha z, \alpha q) \in Y$ too.

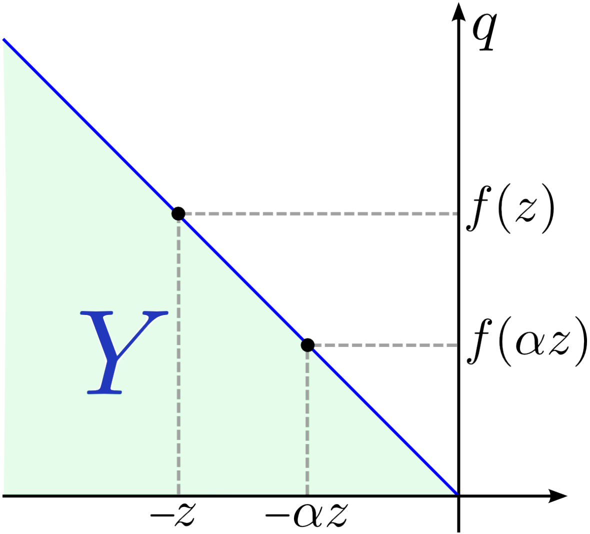
Since it is an 'if and only if' proof, we need to prove the following two statements in the case of single output technology:

$$CRS \Rightarrow H^1 \qquad H^1 \Rightarrow CRS$$

Graphic intuition



Solution: graphical support



Solution I

$CRS \Rightarrow H^1$, part I.

1. Take $z \in \mathbb{R}_+^{L-1}$, such that $(-z, f(z)) \in Y$. Being a single-output case, z is defined over $L-1$ goods and is a vector of inputs only. Take a $\alpha > 0$.
2. The definition of CRS implies that if $y \in Y$, then for any $\alpha > 0$, $\alpha y \in Y$.
3. Then we can say that the point with coordinates $(-\alpha z, \alpha f(z)) \in Y$.
4. From this, we deduce that $\alpha f(z) \leq f(\alpha z)$: if the point above is in Y, it must have a vertical coordinate $(\alpha f(z))$ lower or equal than the maximum amount that is possible to produce given those inputs $(f(\alpha z))$;
5. Hence, we have $\alpha f(z) \leq f(\alpha z)$

□

Solution II

$CRS \Rightarrow H^1$, part II.

1. We can repeat the same argument with a different vector and a different constant;
2. so let's take a vector $\alpha z \in \mathbb{R}_+^{L-1}$ and a constant $\frac{1}{\alpha} > 0$.
3. we have by definition that $(-\alpha z, f(\alpha z)) \in Y$;
4. if *CRS* holds, it must be true that $(-\frac{1}{\alpha}\alpha z, \frac{1}{\alpha} f(\alpha z)) \in Y$.
5. but from this we can derive that $\frac{1}{\alpha} f(\alpha z) \leq f(\frac{1}{\alpha}\alpha z)$
6. that can be simplified, multiplying both sides for α, to $f(\alpha z) \leq \alpha f(z)$.
7. By combining $\alpha f(z) \leq f(\alpha z)$ and $f(\alpha z) \leq \alpha f(z)$, we get the result that $f(\alpha z) = \alpha f(z)$, Q.E.D.

□

Solution III

Proof.
$H^1 \Rightarrow CRS$

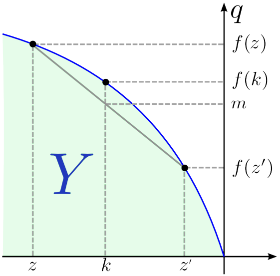
1. Now we have to prove the converse of the proposition proved above.
2. We have to prove hence that if $f(\alpha z) = \alpha f(z) \Rightarrow Y$ satisfies CRS.
3. Take a production plan $(-z, q) \in Y$. Since it is feasible, it must be $q \leq f(z)$.
4. Since q is just a number, it must be that $\alpha q \leq \alpha f(z)$;
5. But we assumed homogeneity of degree one, which means $\alpha f(z) = f(\alpha z)$:
6. hence $\alpha q \leq f(\alpha z)$.
7. Now take another point $(-\alpha z, f(\alpha z)) \in Y$. The point is in Y by the definition of production function.
8. Since $(-\alpha z, f(\alpha z)) \in Y$ and $\alpha q \leq f(\alpha z)$, we get $(-\alpha z, \alpha q) \in Y$, i.e. CRS.

□

2. MWG 5.B.3: convexity and concavity

Show that for a single-output technology, Y is convex if and only if the production function $f(z)$ is concave.

Graphics



$f(\cdot)$ concave $\rightarrow Y$ convex

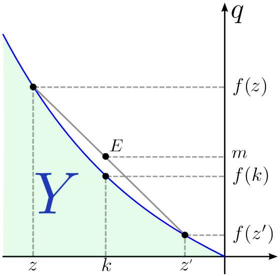
$$k = \alpha z + (1-\alpha)z'$$
$$m = \alpha f(z) + (1-\alpha) f(z')$$

Concavity: $f(k) \geq m$

Convexity: $f(k) \leq m$

Linear: $f(k) = m$

$$E(k,m) \notin Y$$



$f(\cdot)$ convex $\rightarrow Y$ not convex

It's easy to see that intuitively Y is convex if and only if $f(z)$ is concave (or, subcase, linear).

Solution: formal proof, I

Y convex $\Rightarrow f(\cdot)$ concave.

1. Let's take two input vectors

$$z \in \mathbb{R}_+^{L-1} : \ (-z, f(z)) \in Y, \quad z' \in \mathbb{R}_+^{L-1} : \ (-z', f(z')) \in Y$$

2. By convexity of Y (assumed), the linear combination of the two vectors must be in Y

$$(-(\alpha z + (1-\alpha)z'), \alpha f(z) + (1-\alpha) f(z')) \in Y$$

3. Since the above vector is feasible, it must be that

$$\alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha) z')$$

4. ...which is the definition of concavity.

□

Solution: formal proof, II

$f(\cdot)$ concave $\Rightarrow$ Y convex.

1. Let's take two feasible plans, $(-z, q) \in Y$ and $(-z', q') \in Y$, an $\alpha > 0$.
2. This first implies that $q \leq f(z)$ and $q' \leq f(z)$.
3. Hence it must be that

$$\alpha q + (1-\alpha)q' \leq \alpha f(z) + (1-\alpha)f(z')$$

4. By concavity of $f(\cdot)$ (assumed)

$$\alpha f(z) + (1-\alpha)f(z') \leq f(\alpha z + (1-\alpha)z')$$

5. And hence $\alpha q + (1-\alpha)q' \leq f(\alpha z + (1-\alpha)z')$
6. Knowing this, we can build a point that shows that convexity holds: the point

$$(-(\alpha z + (1-\alpha)z'), \alpha q + (1-\alpha)q') \in Y$$

7. that is a linear combination of the plans $(-z, q)$ and $(-z', q')$.

□

3. MWG 5.C.9: profit and supply functions

Derive the profit function $\Pi(p)$ and the supply function (or correspondence) $y(p)$ for the following three single-output technologies, whose production functions $f(z)$ are:

- $f(z) = \sqrt{z_1 + z_2}$
- $f(z) = \sqrt{\min\{z_1, z_2\}}$
- $f(z) = (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}}$, with $\rho = 1$

$f(z) = \sqrt{z_1 + z_2}$: boundary optimum, I

- In this case, non-negativity constraint binds. We are facing a boundary optimum.
- FOCs are not informative. We must hence consider cost minimisation, retrieving from there the profit function.
- Let's impose $p = 1$. we have $q = \sqrt{z_1 + z_2}$.
- Since the two inputs are perfect substitutes, the firm will use only the one with the lower price. Hence

if $w_1 > w_2$

$$\text{then } z_1 = 0 \;\Rightarrow\; z_2 = q^2 \;\Rightarrow\; c = w_2 z_2 = w_2 q^2$$

$$\text{then } \Pi(p, w) = \max_q \; q - w_2 q^2 \;\Rightarrow\; \frac{\partial \Pi}{\partial q} = 0 \;\Rightarrow\; q = \frac{1}{2w_2}$$

$$\text{then } \Pi(p, w) = \frac{1}{2w_2} - w_2 \frac{1}{4w_2^2} = \frac{1}{4w_2}$$

$f(z) = \sqrt{z_1 + z_2}$: boundary optimum, II

if $w_1 = w_2$

$$\text{then } z_1 + z_2 = q^2 \;\Rightarrow\; c = w_2 z_2 = w_2 q^2$$

$$\text{then } \Pi(p, w) = \frac{1}{4w_2} \text{ and } q = \frac{1}{2w_2} \text{ as before}$$

if $w_1 < w_2$

$$\text{then } z_2 = 0 \;\Rightarrow\; z_1 = q^2 \;\Rightarrow\; c = w_1 q^2$$

$$\text{then } \Pi(p, w) = \frac{1}{4w_1} \text{ and } q = \frac{1}{2w_1}$$

Summing up

$$\Pi = \begin{cases} \dfrac{1}{4w_1} & \text{if } \; w_1 < w_2 \\ \dfrac{1}{4w_2} & \text{if } \; w_1 \geq w_2 \end{cases}$$

$f(z) = \sqrt{z_1 + z_2}$: boundary optimum, III

The supply function $y(p, w)$, assuming $p = 1$, is given by

$$y(w) = \begin{cases} \left\{ (0, -\frac{1}{4w_2}, \frac{1}{2w_2}) \right\} & \text{if } \; w_1 > w_2 \\ \left\{ (-z_1, -z_2, \frac{1}{2w_2}), z_1, z_2 \geq 0, z_1 + z_2 = \frac{1}{4w_{1,2}} \right\} & \text{if } \; w_1 = w_2 \\ \left\{ (-\frac{1}{4w_1}, 0, \frac{1}{2w_1}) \right\} & \text{if } \; w_1 < w_2 \end{cases}$$

$f(z) = \sqrt{\min\{z_1, z_2\}}$: non differentiable, I

- The function is not differentiable. But we recognise Leontieff function:
- at optimum, we will have $z_1 = z_2 = q^2$. Hence

$$c = w_1 q^2 + w_2 q^2 = q^2(w_1 + w_2)$$
$$\Pi = q - q^2(w_1 + w_2)$$
$$\frac{\partial \Pi}{\partial q} = 0 \Rightarrow q = \frac{1}{2(w_1 + w_2)}$$
$$z_1 = z_2 = \frac{1}{4(w_1 + w_2)^2} \Rightarrow \Pi = \frac{1}{4(w_1 + w_2)}$$

$$y(p) = \left(-\frac{1}{4(w_1 + w_2)^2}, -\frac{1}{4(w_1 + w_2)^2}, \frac{1}{2(w_1 + w_2)}\right)$$

$f(z) = (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}}$, when $\rho = 1$

If $\rho = 1$, then $f(z) = z_1 + z_2$, and non-negativity constraint binds. Assuming $p = 1$, if input prices are higher than p it is optimal not to produce; in the opposite case production is $+\infty$. Solutions are

$$\Pi(w) = \begin{cases} 0 & \text{if } \min\{w_1, w_2\} \geq 1 \\ \infty & \text{if } \min\{w_1, w_2\} < 1 \end{cases}$$

$$y(w) = \begin{cases} \{0\} & \text{if } \min\{w_1, w_2\} \geq 1 \\ \infty & \text{if } \min\{w_1, w_2\} < 1 \\ \{\alpha(-1, 0, 1) : \alpha \geq 0\} & \text{if } 1 = w_1 < w_2 \\ \{\alpha(0, -1, 1) : \alpha \geq 0\} & \text{if } w_1 > w_2 = 1 \\ \{\alpha(-z_1, -z_2, 1) : \alpha, z_1, z_2 \geq 0, z_1 + z_2 = 1\} & \text{if } w_1 = w_2 = 1 \end{cases}$$

4. Cobb-Douglas production function: all you ever wanted to know

Consider a Cobb-Douglas production function, $f(z) = z_1^{\alpha} z_2^{\beta}$ with $\alpha, \beta > 0$. For the three cases in which $\alpha + \beta <, =, > 1$:

- draw Y (in 3d), marginal and average product, and the rate of technical substitution (in 2d);
- solve the profit maximisation and the cost minimisation problems;
- find conditional factor demand functions;
- find supply functions (correspondences);
- find cost functions.

Returns to scale

- First, let's consider that Cobb-Douglas is homogeneous of degree $\alpha + \beta$:
- let's take $f(z) = z_1^\alpha z_2^\beta$, and a positive constant $k > 0$.
- then by applying the definition of homogeneity,

$$f(kz) = kz_1^\alpha kz_2^\beta = k^{\alpha+\beta} z_1^\alpha z_2^\beta = k^{\alpha+\beta} f(kz)$$

- hence, if $\alpha + \beta = 1$, we have homogeneity of degree one and CRS (see proof);
- if $\alpha + \beta < 1$ we have decreasing returns to scale;
- if $\alpha + \beta > 1$ we have inreasing returns to scale.

For graphics of the three cases, see the file `CD.pdf`, (courtesy Peter Fuleky, U.Washington) on ARIEL. It depicts production sets Y, isoquants, marginal product, returns to scale.
In the case of single output - single input technology, see the file `GraphsProduction.pdf` on ARIEL.

Returns to scale and problems

- The profit function is not always defined for any production function.
- In the case of CRS, profit is either zero or indeterminate:
  - Consider $f(x) = x$. Then Profit is defined as $px - wx$.
  - It is clear that if $p > w$, the marginal profit is positive and constant, leading to infinite production and profit;
  - On the other side, if $p < w$, profit is negative and hence it is optimal not to produce;
  - Finally, if $p = w$, production is indeterminate as all production levels will imply zero profit.
- In the case of IRS, profit is always infinite:
  - since returns are increasing, cost is decreasing with output;
  - since marginal revenue is constant (price-taking), profit $(MR - MC)$ will be always increasing in q;
  - it will be then optimal to produce an infinite amount, with infinite profit.

what will we do?

- Analyse CRS on its own;
- Use DRS (profit max well defined) to find Π and $z(p, w)$;
- Use IRS (cost min well defined) to find $c(p, q)$ and conditional demand $z(p, q)$

CRS: $\alpha + \beta = 1$, I

As in all cases involving CRS, profit maximisation is not well defined. We hence turn to the cost minimisation problem

$$c = \min w_1 z_1 + w_2 z_2 \text{ s.t.: } \; q = z_1^{\alpha} z_2^{\beta}$$

Which has FOCs

$$FOC = \begin{cases} w_1 - \lambda \alpha z_1^{\alpha-1} z_2^{\beta} = 0 \\ w_2 - \lambda \beta z_1^{\alpha} z_2^{\beta-1} = 0 \end{cases}$$

which, imposing $\alpha + \beta = 1$ can be solved to yield:

$$z_1 = q \left(\frac{\alpha w_2}{\beta w_1} \right)^{\beta} \quad , \quad z_1 = q \left(\frac{\beta w_1}{\alpha w_2} \right)^{\alpha}$$

Then, imposing $q = 1$ and plugging, we get unit cost to be:

$$c = \left(\frac{w_1}{\alpha} \right)^{\alpha} \left(\frac{w_2}{\beta} \right)^{\beta}$$

CRS: $\alpha + \beta = 1$, II

Hence, as in all cases involving CRS, production is:

$$y(w) = \begin{cases} 0 & \text{if } c > p \\ \infty & \text{if } c < p \\ \forall q & \text{if } c = p \end{cases}$$

DRS: $\alpha + \beta < 1$, I

In this case, profit maximisation is well specified with an interior solution, and is

$$\Pi = \max \; p z_1^{\alpha} z_2^{\beta} - w_1 z_1 - w_2 z_2$$

Which has FOCs

$$FOC = \begin{cases} w_1 = \alpha z_1^{\alpha-1} z_2^{\beta} \\ w_2 = \beta z_1^{\alpha} z_2^{\beta-1} \end{cases}$$

which can be solved, imposing $1 - \alpha - \beta = \gamma$ to yield

$$z_1 = \left(\frac{\alpha}{w_1}\right)^{\frac{1-\beta}{\gamma}} \left(\frac{\beta}{w_2}\right)^{\frac{\beta}{\gamma}} p^{\frac{1}{\gamma}} \;\; , \;\; z_2 = \left(\frac{\alpha}{w_1}\right)^{\frac{\alpha}{\gamma}} \left(\frac{\beta}{w_2}\right)^{\frac{1-\alpha}{\gamma}} p^{\frac{1}{\gamma}}$$

DRS: $\alpha + \beta < 1$, II

The supply function can then be computed by plugging $q = f(z(p, w))$:

$$q = z_1^{\alpha} z_2^{\beta} = p^{\frac{\alpha+\beta}{\gamma}} \left(\frac{\alpha}{w_1}\right)^{\frac{\alpha}{\gamma}} \left(\frac{\beta}{w_2}\right)^{\frac{\beta}{\gamma}}$$

And the profit function is computed by plugging $q(p, w)$ into Π:

$$\Pi = p \cdot q - w \cdot z = \gamma p^{\frac{1}{\gamma}} \left(\frac{\alpha}{w_1}\right)^{\frac{\alpha}{\gamma}} \left(\frac{\beta}{w_2}\right)^{\frac{\beta}{\gamma}}$$

IRS: $\alpha + \beta > 1$, I

In this case profit maximisation is not defined, always giving infinite profits. Cost minimisation is quite standard instead.

$$c = \min w_1 z_1 + w_2 z_2 \text{ s.t.: } \; q = z_1^{\alpha} z_2^{\beta}$$

Which has FOCs

$$FOC = \begin{cases} w_1 - \lambda \alpha z_1^{\alpha-1} z_2^{\beta} = 0 \\ w_2 - \lambda \beta z_1^{\alpha} z_2^{\beta-1} = 0 \end{cases}$$

Which can be solved, without imposing any constraint on $\alpha + \beta$, to yield:

$$z_1 = q^{\dfrac{1}{\alpha+\beta}} \left(\frac{\alpha w_2}{\beta w_1}\right)^{\dfrac{\beta}{\alpha+\beta}} \quad , \quad z_1 = q^{\dfrac{1}{\alpha+\beta}} \left(\frac{\beta w_1}{\alpha w_2}\right)^{\dfrac{\alpha}{\alpha+\beta}}$$

IRS: $\alpha + \beta > 1$, II

To get the cost function, ust plug the $z(p, q)$ values in the minimand:

$$c = w_1 z_1 + w_2 z_2 = (\alpha + \beta) q^{\frac{1}{\alpha + \beta}} \left(\frac{w_1}{\alpha}\right)^{\frac{\alpha}{\alpha + \beta}} \left(\frac{w_2}{\beta}\right)^{\frac{\beta}{\alpha + \beta}}$$

And, of course, the profit function is the same as the one derived in the case in which $\alpha + \beta < 1$ was imposed.